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Growth of Betti Numbers of Modules over Generalized Golod Rings

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1. INTRODUCTION

Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, and let M be a finitely generated R-module. A resolution

$$\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to M \to 0$$

of M by free modules of finite rank, such that $\partial_n(F_n) \subseteq \mathfrak{m} F_{n-1}$ for all $n \geq 1$, is called a *minimal free resolution*. It is known to be unique up to isomorphism of complexes.

The nth Betti number of M over R is the integer $b_n = \operatorname{rank} F_n$. The sequence $\{b_n\}_{n \geq 0}$ is called the *Betti sequence* of M. It is said to have *strongly exponential growth*, cf. [3, p. 34], if there are real numbers $1 < A \leq B$ such that the inequalities

$$A^n \leq b_n \leq B^n$$

hold for all sufficiently large n.

The following questions are raised by Avramov:

Problem 1 [2, *Sect*. 5]. Is the Betti sequence eventually nondecreasing?

Problem 2 [3, (1.2)]. Is the growth of the Betti sequence either polynomially bounded or strongly exponential?

It is known that there always exists a positive real number B such that $b_n \leq B^n$ for all n, cf. [3, (2.5)], hence the *radius of convergence* ρ of the

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Poincaré series $P_M^R(t) = \sum_{n \geq 0} b_n t^n$ is positive. Avramov proposes in [5] to use the reciprocal value $1/\rho$ as a measure for the asymptotic growth of Betti sequence, and calls it the *curvature* of M:

$$\operatorname{curv}_R M = \limsup_{n \to \infty} \sqrt[n]{b_n}\,.$$

When $P_M^R(t)$ represents a rational function, $\operatorname{curv}_R M \leq 1$ if and only if $\{b_n\}_{n \geq 0}$ is polynomially bounded, cf. [3, (2.3)]. The most general class of local rings for which $P_M^R(t)$ is known to be rational for each M is given by *the generalized Golod rings* introduced in [4] (the definition is recalled in Section 2). They include the complete intersections. Rationality is an important property, as Anick [1] has constructed rings for which $P_k^R(t)$ is not rational.

In this paper we prove:

THEOREM. *Let M be a finitely generated module over a generalized Golod ring R. If* $\operatorname{curv}_R M = \gamma > 1$, *then*

$$\lim_{n \to \infty} \frac{b_n}{\gamma^n} > 0.$$

Together with [3, (2.3)], the following corollary provides a positive answer to Problem 2 for modules over generalized Golod rings.

COROLLARY. *For R and M as above,* $\lim_{n \to \infty}(b_{n+1}/b_n) = \gamma$. *In particular, the Betti sequence is eventually increasing and grows strongly exponential.*

Examples of generalized Golod rings include Golod rings, rings of small embedding codimension or small linkage number (Avramov, Jacobsson, Kustin, Miller, Palmer), and some rings defined by Huneke−Ulrich ideals (Kustin). For these examples, the conclusion of the theorem is known and can be found in [13, 14, 10, 11] where each case required a separate argument and used the explicit form of the denominator of $P_M^R(t)$. In these special cases, Problem 1 is known to have a positive answer also when $\operatorname{curv}_R M = 1$, but requires a completely different approach. It is still open for generalized Golod rings.

2. GROWTH OF BETTI NUMBERS

1. *Tate Resolutions*. We recall Tate's process [15] of killing cycles by adjoining a set X of exterior and divided powers variables.

Choose a set of exterior variables X_1 such that $\partial(X_1)$ is a minimal set of generators of $\mathfrak{m}$. For $q \geq 2$, choose X_q such that $\partial(X_q)$ is a basis of the

k-vector space $H_{q-1}(R\langle X_{<q}\rangle)$, where $X_{<q} = \{x \in X : |x| < q\}$. Here $x \in X_q$ is a divided powers variable if q is even, and an exterior variable if q is odd. Iterating this process, one gets a DG algebra of the form $R\langle X\rangle$ with $H(R\langle X\rangle) \cong k$.

Gulliksen [6] and Schoeller [12] have proved that $R\langle X\rangle$ is a minimal free resolution of k. For $q \geq 1$, the construction yields

$$H_0\big(R\langle X_{\leq q}\rangle\big) \cong k; \tag{1}$$

$$H_i\big(R\langle X_{\leq q}\rangle\big) = 0 \qquad \text{for } 0 < i < q. \tag{2}$$

LEMMA. *For $q \geq 2$, we have*

$$H_{q+1}\big(R\langle X_{\leq q}\rangle\big) \cong H_{q+1}\big(R\langle X_{\leq q+1}\rangle\big). \tag{3}$$

Proof. Set $U = R\langle X_{\leq q}\rangle$. If $H_q(U) = 0$, then there is nothing to prove since from the construction one has $R\langle X_{\leq q}\rangle = R\langle X_{\leq q+1}\rangle$. Hence we assume that $H_q(U) \neq 0$. Let z be a cycle in U such that $0 \neq \operatorname{cls}(z) \in H_q(U)$ and adjoins a variable x such that $\partial(x) = z$.

If q is even, then $U\langle x : \partial(x) = z\rangle$ is the exterior algebra over U of the free U-module on a generator x of degree $q+1$. From the proof of [15, Theorem 2], there is a long exact sequence on homology

$$\begin{aligned}&\to H_1(U) \to H_{q+1}(U) \to H_{q+1}(U\langle x\rangle) \to H_0(U) \xrightarrow{\alpha} H_q(U)\\ &\to H_q(U\langle x\rangle) \to H_{-1}(U) \to \cdots.\end{aligned}$$

By (1) and (2), $H_1(U) = 0 = H_{-1}(U)$, $H_0(U) \cong k$, and $\alpha(1) = \operatorname{cls}(z) \neq 0$, one concludes that $H_{q+1}(U) \cong H_{q+1}(U\langle x\rangle)$.

If q is odd, then $U\langle x : \partial(x) = z\rangle$ is the graded free U-module with basis $\{x^{(i)} : |x^{(i)}| = i(q+1)\}_{i \geq 0}$. In this case, from *loc. cit.* there is a long exact sequence

$$\begin{aligned}H_1(U\langle x\rangle) &\to H_{q+1}(U) \to H_{q+1}(U\langle x\rangle) \to H_0(U\langle x\rangle) \xrightarrow{\alpha} H_q(U)\\ &\to H_q(U\langle x\rangle) \to H_{-1}(U\langle x\rangle).\end{aligned}$$

By (1) and (2), $H_1(U\langle x\rangle) = 0 = H_{-1}(U\langle x\rangle)$, $H_0(U\langle x\rangle) \cong k$, and $\alpha(1) = \operatorname{cls}(z) \neq 0$, one concludes that $H_{q+1}(U) \cong H_{q+1}(U\langle x\rangle)$.

Iterating this process, we obtain (3). ∎

2. *Halperin's Theorem.* The cardinality e_q of the set X_q is known to be independent of the construction. It is called the qth deviation of R. In [9, Theorem B], Halperin proves that if R is not a complete intersection,

then $e_q \neq 0$ for all $q \geq 1$, that is,

$$H_q\big(R\langle X_{\leq q}\rangle\big) \neq 0 \qquad \text{for } q \geq 1. \tag{4}$$

3. *Gulliksen's Theorem*. Set $U = R\langle X_{\leq q}\rangle$. It is proved in [8, Theorem 2], cf. also [4, (1.4)] that the power series

$$\sum_{i \geq 0} a_i t^i, \qquad \text{where } a_i = \dim_k H_i(U)$$

represents a rational function

$$h(t) = \frac{u(t)}{v(t)} \qquad \text{with } v(t) = \prod_{0 \leq 2i \leq q} (1 - t^{2i})^{e_{2i}} \text{ and } u(t) \in \mathbb{Z}[t]. \tag{5}$$

4. *Avramov's Theorem*. Let $(R, \mathfrak{m}, k)$ be a local ring, let U be a DG R algebra with $H_0(U) \cong k$, and let $B = \{h_i\}_{i \geq 1}$ be a homogeneous basis of the k-vector space $\operatorname{Ker}(\varepsilon_{H(U)} : H(U) \to k)$. The DG algebra U is called *Golod* if there is a function μ from the set of finite sequences of elements of B to $\operatorname{Ker}(\varepsilon_U : U \to k)$ such that

$$\mu(h_i) = z_i \qquad \text{with } \partial(z_i) = 0 \text{ and } \operatorname{cls}(z_i) = h_i;$$

$$\partial\mu\big(h_{i_1}, \dots, h_{i_n}\big) = \sum_{j=1}^{n-1} \overline{\mu\big(h_{i_1}, \dots, h_{i_j}\big)}\mu\big(h_{i_{j+1}}, \dots, h_{i_n}\big),$$

where $\bar{a}$ stands for $(-1)^{|a|+1}a$.

Let $R\langle X\rangle$ denote the Tate resolution described above and set $X_{\leq q} = \{x \in X : |x| \leq q\}$. In [4, (1.7)], a ring R is called *generalized Golod* of level $\leq q$ if the DG algebra $U = R\langle X_{\leq q}\rangle$ is Golod. It is proved in [4, (1.4)] that for each M there exists a polynomial $p_M(t) \in \mathbb{Z}[t]$ such that

$$P_M^R(t) = \frac{p_M(t)}{v(t) \cdot w(t)} \tag{6}$$

with $v(t)$ as in (5) and

$$w(t) = 1 + t - t \cdot h(t), \tag{7}$$

where $h(t)$ is as in (5).

To prove our main theorem, we need the following property of

5. *Rational Functions*. Let $\sum_{n \geq 0} b_n t^n$ be the Taylor expansion of a rational function $f(t)$. Assume that $b_n \geq 0$ for all n, and that the radius of convergence ρ of $f(t)$ satisfies $\rho \leq 1$. It is well known and easy to see that

the non-negativity of b_n's implies that ρ is a pole of $f(t)$. Furthermore, if the order d of the pole at ρ of $f(t)$ is strictly bigger than the orders of its other poles on the circle $|z| = \rho$, then

$$\lim_{n \to \infty} b_n \frac{\rho^n}{n^{d-1}} > 0. \tag{8}$$

This is proved in [13, (2.2)].

6. *Proof of the Theorem*. The idea is to study the rational function $w(t)$ from (6). First, by (5) and (7), $w(t)$ converges inside the unit circle. On the other hand, because $P_M^R(t)$ converges in a circle of radius $\rho = \gamma^{-1} < 1$, the function $w(t)$ converges at ρ. By (6), the radius of convergence ρ has to be a root of $v(t) \cdot w(t)$, hence $w(\rho) = 0$ by (5). Thus, ρ is a root of $w(t)$.

Furthermore, under the hypothesis of the theorem R cannot be a complete intersection since in that case $\gamma = \operatorname{curv}_R M \leq 1$ by a result of Gulliksen [7]. From (1), (2), $h(t)$ has the Taylor expansion of the form $1 + \sum_{i \geq q} a_i t^i$ with $a_i = \dim_k H_i(U) \geq 0$. Thus, by (3), (4), and (7), the function $w(t)$ has the Taylor expansion

$$w(t) = 1 - a_q \cdot t^{q+1} - a_{q+1} \cdot t^{q+2} - \dots, \qquad \text{with all } a_i \geq 0, a_q \neq 0, a_{q+1} \neq 0. \tag{9}$$

Since the derivative of the analytic function $w(t)$ has the Taylor expansion

$$w'(t) = -(q+1) \cdot a_q \cdot t^q - (q+2) \cdot a_{q+1} \cdot t^{q+1} - \dots,$$

we also have $w'(\rho) < 0$. Thus, ρ is a simple root of $w(t)$.

Let ξ be a complex number with $|\xi| = 1$.

If $\xi = -1$, then by (9) and the fact that either $q+1$ or $q+2$ is odd, we have $w(\xi\rho) = w(-\rho) > w(\rho) = 0$.

If ξ is not real, then either ξ^{q+1} or ξ^{q+2} is not real. If ξ^{q+1} is not real, we write $w(t)$ as $h_1(t) - h_2(t)$, where $h_1(t) = 1 - a_q t^{q+1}$ and $h_2(t) = a_{q+1} t^{q+2} + \cdots$. An easy computation shows that

$$|h_1(\rho\xi)| > |h_1(\rho)| = |h_2(\rho)| \geq |h_2(\rho\xi)|. \tag{10}$$

If ξ^{q+2} is not real, then we choose $h_1(t) = 1 - a_{q+1} t^{q+2}$ and $h_2(t) = a_q t^{q+1} + a_{q+2} t^{q+3} + \cdots$. The inequalities (10) hold.

The three cases above cover all possibilities, hence $w(\rho\xi) \neq 0$. Thus, ρ is the only root of $w(t)$ on the circle of the radius of convergence of $P_M^R(t)$.

It follows that we can apply (8) with $d = 1$. This gives the assertion of the Theorem. ∎

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